A study of non-Boussinesq effect on transition of thermally induced flow in a square cavity

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The low-Mach number approximation is considered to be less restrictive than the Boussinesq approximation. The former represents well the flow behavior in enclosures with relatively large temperature difference, and therefore permits a better understanding of thermally driven flows. However, as the flow becomes chaotic or weakly turbulent, it becomes uncertain whether the low-Mach number approximation can properly describe the transition behavior of the flow. In the present study, the behavior of transitional thermally driven flow in a two-dimensional (2-D) differentially heated square cavity filled with a gas has been numerically investigated in cases where the temperature difference increases. The exact governing equations, including the equation of state for ideal gas, are used to calculate the initial phase to transition, and the results are compared in detail with those obtained using the Boussinesq-approximated and low-Mach number equations. The applicability of these approximated governing equations has been discussed. It turned out that beyond a certain Rayleigh number, even the low-Mach number approximation may not always predict the transition behavior of the flow.

Keywords: natural convection; thermally driven flow; enclosure; instability; bifurcation; governing equation

Introduction

It is well recognized that thermal convection (i.e., thermally driven flow) in differentially heated enclosures has important technological applications, such as solar energy collectors, electronic equipment, cooling of nuclear reactors, ventilation of rooms, and crystal growth in materials processing. For thermal convection in an enclosure with complex and realistic conditions, a detailed research review has been made by Fusegi and Hyun (1994), considering spatial and temporal variations of thermal boundary condition, variable property effects, and three-dimensionalities.

In general, the numerical treatment of this problem has been based on the Boussinesq approximation (Boussinesq 1903). However, this approximation is limited, in principle, to very small temperature difference, and thus its use for analysis does not always give the physically exact behavior of thermal convection under the practical conditions.

Recently, owing to the advent of powerful computers and better numerical algorithms, a set of governing equations called "low-Mach number approximation" has gained increasing usage. These equations are less restrictive than the Boussinesq-approximated equations and can be used to simulate thermally driven flows with a large temperature difference (Rehm and Baum 1978; Paolucci 1982; Fröhlich et al. 1992; Horibata 1992). There are several analyses of thermal convection in a square cavity using these equations at high-Rayleigh number (Paolucci and Chenoweth 1989; Paolucci 1990; Le Quéré 1992). However, as the flow becomes chaotic or weakly turbulent (relatively large velocity, temperature, and pressure fluctuations), it becomes uncertain whether the low-Mach number approximation can properly describe the transition behavior of the flow.

In the present study, the exact governing equations including the state equation for ideal gas are used to calculate the initial phase to transition of thermal convection in a square cavity with increasing temperature differences. The results are compared in detail with those obtained using the Boussinesq-approximated and low-Mach number equations. To the authors' knowledge, the examination regarding the limit of applicability of the low-Mach number equations is not available in the literature. Therefore, attention is focused on the applicability of these approximated governing equations. The results for different Rayleigh numbers and temperature differences parameters are illustrated with time histories, power spectra, mean temperature, mean velocity, and intensities of velocity and temperature fluctuations.

Problem statement and governing equations

The problem treated in this study is thermal convection in a two-dimensional (2-D) square cavity of size L. As shown in Figure 1, the left and right side walls of the cavity are isothermal at respective temperatures of T_h and T_c ($T_h > T_c$), and the bottom and top walls are adiabatic. The working fluid is a gas,

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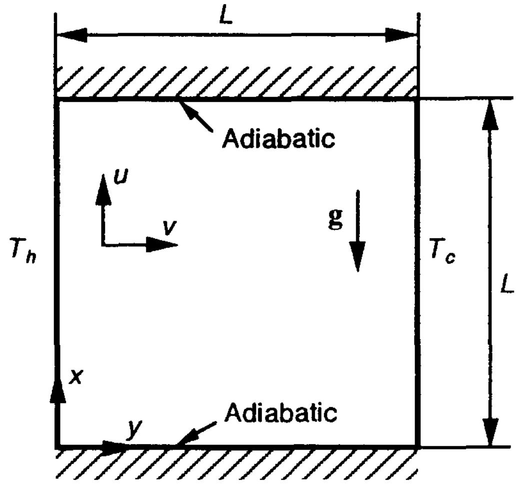

Figure 1 Schematic of a square cavity

and all fluid properties except density are assumed to be constant, with attention being focused on the buoyancy effect generated by the density variation. If the density variation cannot be properly modeled, it may become worthless to model other fluid properties. Then, the governing equations for conservation of mass, momentum, and energy with the ideal gas law become the following:

$$\mathrm{D}\rho/\mathrm{D}\tau + \rho(\nabla \cdot \mathbf{v}) = 0 \quad (1)$$

$$\rho \mathrm{D}\mathbf{v}/\mathrm{D}\tau = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \quad (2)$$

$$\rho c_p \mathrm{D}T/\mathrm{D}\tau = \lambda \nabla^2 T \quad (3)$$

$$p = \rho R T \quad (4)$$

The compressibility term in Equation 2 and the viscous dissipation and the pressure terms in Equation 3 are neglected, because they are several orders of magnitude smaller than the other terms for a quasi-steady closed system under the condition of uniform wall temperature (no significant difference has been observed between the results calculated with and without these small-order terms). By normalizing these equations with the reference length L, the temperature difference ΔT_w and physical properties, four dimensionless parameters governing this system have been noticed: the Galilei number Ga, the Prandtl number Pr, the temperature difference parameter $\beta\,\Delta T_w$, and the coefficient $\gamma \mathrm{Ma}^2$ (the Mach number is defined as $\mathrm{Ma} = (g\beta\,\Delta T_w\,L)^{1/2}/c$, where c is the velocity of sound). In the present study, these governing equations are referred to as the exact governing equations.

On the other hand, the low-Mach number equations (LMN) are obtained from the expansion of all variables, such as velocity, pressure, density, and temperature in a series of small Mach numbers. Then, the pressure included in the energy equation, which is neglected in Equation 3, is approximated by a thermodynamic pressure dependent only on time. This pressure is determined by the space integral of the thermal energy equation for instantaneous density and temperature. However, because the pressure term is very small under the quasi-steady condition, as mentioned above, the variation of density can be expressed as a function of temperature independent of pressure, and thus, the state equation may be expressed as follows:

$$\rho T = \mathrm{const} \quad (5)$$

Consequently, Equations 1–3 and 5 correspond to the low-Mach number equations. The three dimensionless parameters governing this system are the Galilei number Ga, the Prandtl number Pr, and the temperature difference parameter $\beta\,\Delta T_w$. As long as Equation 5 is strictly maintained (e.g., steady laminar flow), the low-Mach number equations can describe the correct flow behavior.

Most of the existing numerical works concerning thermal convection are based on the governing equations with the Boussinesq approximation. These equations are expressed as follows:

$$\nabla \cdot \mathbf{v} = 0 \quad (6)$$

$$\mathrm{D}\mathbf{v}/\mathrm{D}\tau = -\nabla p^*/\rho + \nu \nabla^2 \mathbf{v} - \mathbf{g}\beta(T - T_o) \quad (7)$$

$$\mathrm{D}T/\mathrm{D}\tau = \alpha \nabla^2 T \quad (8)$$

Notation

c	velocity of sound, $(\gamma R T_0)^{1/2}$
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
$\mathrm{D}/\mathrm{D}\tau$	substantial derivative
E_u	vertical velocity fluctuation spectra
f	frequency of fluctuation
$\mathbf{g}$	gravitational vector
g	gravitational acceleration
Ga	Galilei number, gL^3/ν^2
Gr	Grashof number, Ga $\beta\,\Delta T_w$
L	reference length (square cavity side length)
Ma	Mach number, $(g\beta\,\Delta T_w\,L)^{1/2}/c$
p	pressure
Pr	Prandtl number, $\mu c_p/\lambda$
R	gas constant
Ra	Rayleigh number, Gr Pr
T	temperature
T_c, T_h	temperatures of cooled and heated isothermal side walls
T_0	reference temperature
$\mathbf{v}$	velocity vector
u, v	vertical and horizontal velocity components
x, y	coordinates in vertical and horizontal directions

Greek

α	thermal diffusivity
β	coefficient of thermal expansion
ΔT_w	temperature difference, $T_h - T_c$
λ	thermal conductivity
γ	ratio of specific heats, c_p/c_v
μ	viscosity
ν	kinematic viscosity
ρ	density
ρ_0	reference density
τ	time

Superscripts

$\overline{(\ )}$	time-averaged value
$(\ )'$	fluctuating component

The pressure p^* in Equation 7 represents the pressure changing with fluid motion. These equations are derived by assuming that only the density related to the body force varies proportionally with temperatures, while other properties are kept constant. Thus, the mass conservation equation results in a simple form, and theoretical treatment becomes much easier. Dimensionless parameters relevant to Equations 6–8 are the Grashof number Gr ($= \mathrm{Ga}\beta\,\Delta T_w$) and the Prandtl number Pr.

Numerical results and discussion

The present study is restricted to the calculation of the initial phase of transition. The cavity is filled with air (Pr = 0.71), and the computations are performed for Rayleigh numbers Ra (= Gr Pr) ranging from 1.6×10^8 to 3.5×10^8, with a fixed cavity size and varying the temperature difference parameter $\beta\,\Delta T_w = 0.05 \sim 0.112$. The coefficient of thermal expansion is evaluated at the reference temperature $T_0 = (T_h + T_c)/2$ as $\beta = 1/T_0$, so that $\beta\,\Delta T_w = 0.112$ corresponds to $\Delta T_w \simeq 34$ K, and the reference temperature $T_0 = 300$ K. The chief aim of the present study is to examine the applicability of the approximated governing equations (namely, the Boussinesq-approximated and low-Mach number equations) by comparing their characteristic solutions with those obtained from the exact governing equations. For solving the governing equations, a well-tested control volume formulation based on the SIMPLE pressure-correction method, second-order central difference scheme for convective and diffusive terms has been used. The domain is covered with nonequidistant grids of 120×120 (x, y), having a concentration of grid lines near the walls and staggered grids for velocities. To check both the accuracy and correct implementation of the 2-D code, steady-state calculations at $\mathrm{Ra} = 10^8$ were performed with the Boussinesq approximation. The results for the Nusselt number and velocity profiles almost agreed with those obtained by Le Quéré (1991) and Henkes and Hoogendoorn (1993). The first instability (occurrence of small oscillation) was noticed at $1.7 \times 10^8 < \mathrm{Ra} < 1.8 \times 10^8$ regardless of which governing equations were used. This is in good agreement with the results reported by Chenoweth and Paolucci (1986) and Henkes (1990).

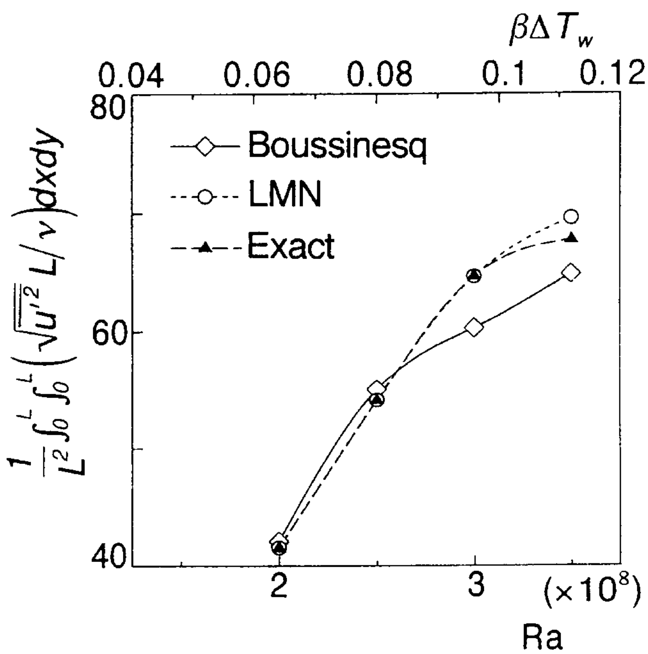

Figure 2 Overall intensities of vertical velocity fluctuations

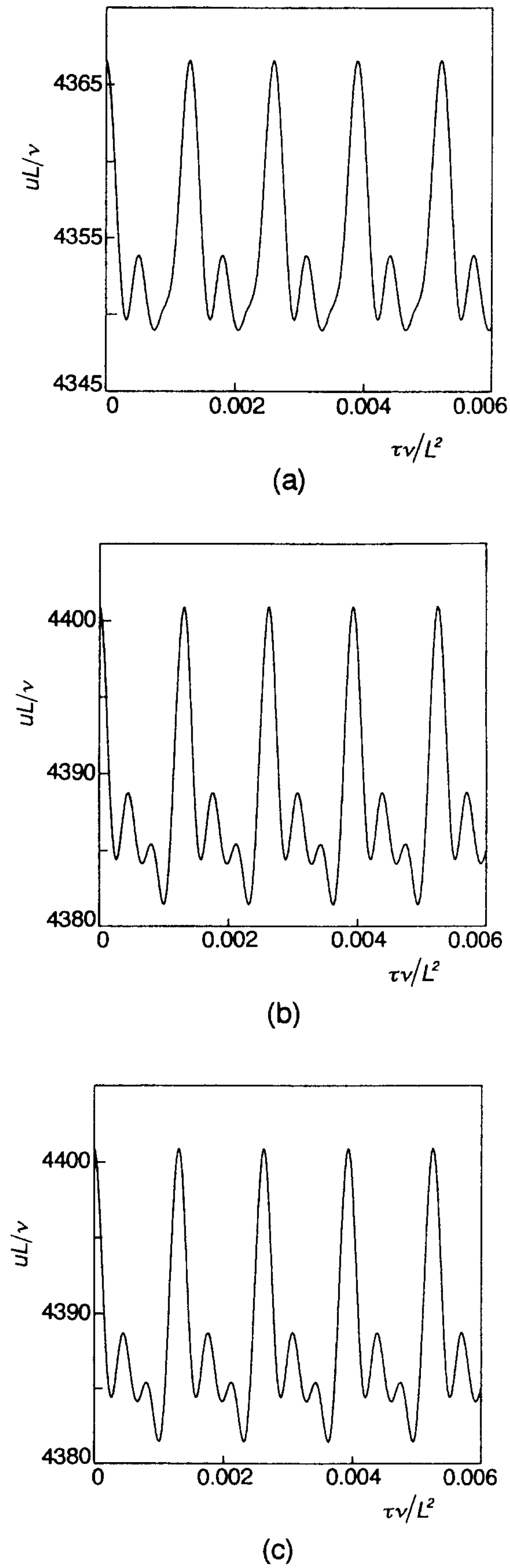

Figure 3 Time histories for maximum vertical velocities at half the cavity height ($\mathrm{Ra} = 2 \times 10^8$); (a) Boussinesq; (b) LMN ($\beta\,\Delta T_w = 0.065$); (c) exact ($\beta\,\Delta T_w = 0.065$, $\gamma \mathrm{Ma}^2 = 10^{-5}$)

To investigate the development of flow oscillation with an increase in the Rayleigh number, the spatial averaged values of the intensities of vertical velocity fluctuations have been plotted in Figure 2. There is no distinction among the solutions obtained from the three different governing equations up to $Ra = 2.5 \times 10^8$, although the result with Boussinesq approximation shows somewhat high intensities. As the Rayleigh number increases beyond 2.5×10^8, high-frequency fluctuations are superposed on low-frequency oscillations, and the flow becomes quasi-periodic, which suggests the occurrence of a so-called second bifurcation (Henkes 1990; Xia et al. 1995). This phenomenon is observed regardless of the system of governing equations. Then, a difference between the exact and Boussinesq-approximated solutions clearly appears; whereas, the intensities obtained from the low-Mach number equations coincide with those of the exact equations up to $Ra = 3 \times 10^8$. However, beyond $Ra = 3 \times 10^8$, the solution given by the low-Mach number approximation deviates gradually from the exact solution.

The periodicity in the vertical velocity maximum observed near the hot vertical wall at half the cavity height for $Ra = 2 \times 10^8$ is shown in Figure 3. It is obvious that solutions of the low-Mach number and exact equations coincide quite well (Figures 3b and 3c), and the dominant low frequency reasonably agrees with that predicted by Henkes (1990). The solution of the Boussinesq-approximated equations shows a basic periodicity similar to that of the exact solution, but the waveform and the value of the mean vertical velocity are different from those of the exact solution (Figure 3a).

At $Ra = 2.5 \times 10^8$, when the second bifurcation generates, the flow becomes quasi-periodic, and the normalized power spectra of the fluctuations of the maximum vertical velocity observed near the hot wall at half the cavity height are characterized with marked peaks at fundamental frequencies, as shown in Figure 4. This quasi-periodic regime is observed for the three different sets of governing equations. However, as the Rayleigh number increases to 3×10^8, these distinct peaks disappear in the power spectrum with the Boussinesq approximation (Figure 5a); whereas, the low-Mach number and exact equations give almost the same spectrum still having three peaks as shown in Figures 5b and 5c. Figure 6 compares the power spectra obtained from the low-Mach number and exact equations at $Ra = 3.5 \times 10^8$. When the low-Mach number approximation is used, the flow can no longer be described by a small number of well-defined characteristic frequencies, showing that the flow is becoming chaotic or weakly turbulent. On the other hand, the solution of the exact equations still gives quasi-periodic characteristics, as shown in Figure 6b. These results indicate that the flow development within the square cavity passes through different dynamical regimes depending on the governing equations, and premature chaotic flows are predicted with the approximated governing equations (i.e., the Boussinesq-approximated and low-Mach number equations) for the Rayleigh number increment. Also, for the same Rayleigh number ($Ra = 3.5 \times 10^8$), an evaluation of the neglected pressure term in Equation 3 has been attempted. It is found that this term is of order 2.5×10^{-4} of the other terms. This supports what we previously mentioned regarding the pressure term.

Figure 7 shows the mean vertical velocity and temperature distributions near the hot and cold vertical walls at half the cavity height calculated with the Boussinesq-approximated, low-Mach number and exact equations at $Ra = 3.5 \times 10^8$. A general aspect is that the mean velocity and temperature in the boundary layers formed near the vertical walls have laminar-like profiles. There is no discrepancy among the results despite the fact that the fluctuation spectra are different form each other, as seen in Figure 5.

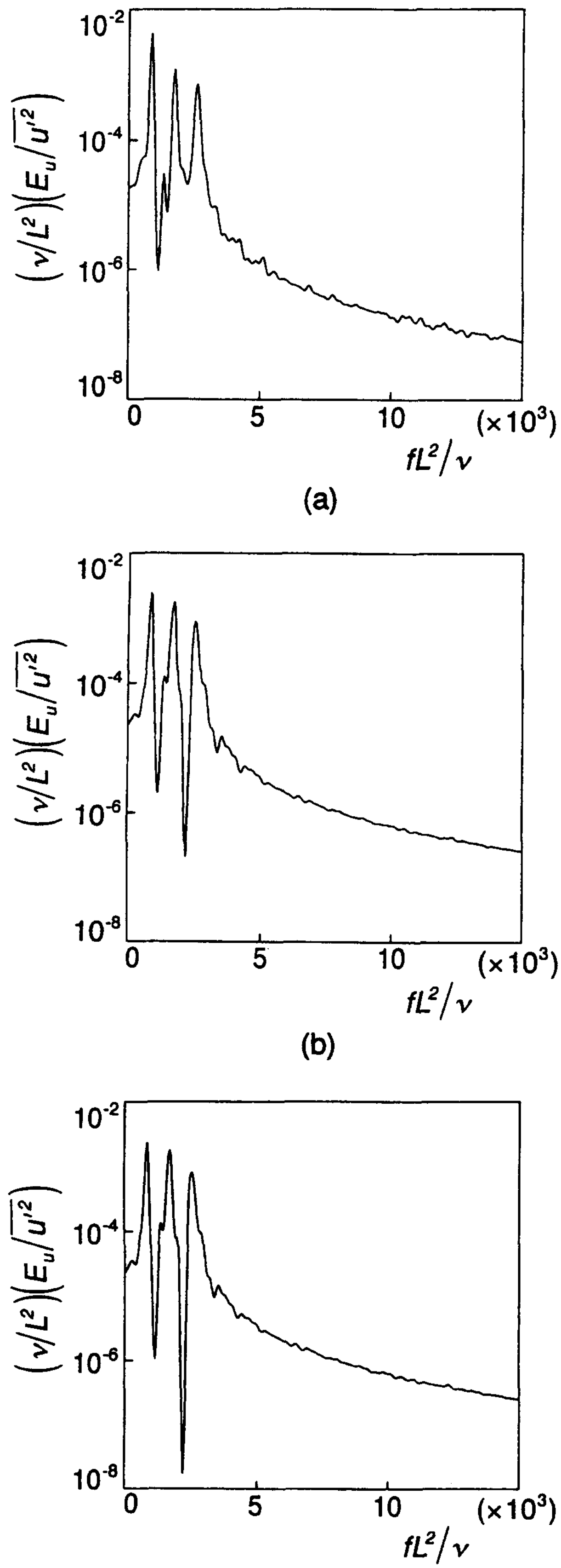

Figure 4 Velocity fluctuation spectra ($Ra = 2.5 \times 10^8$); (a) Boussinesq; (b) LMN ($\beta \Delta T_w = 0.08$); (c) exact ($\beta \Delta T_w = 0.08$, $\gamma Ma^2 = 10^{-5}$)

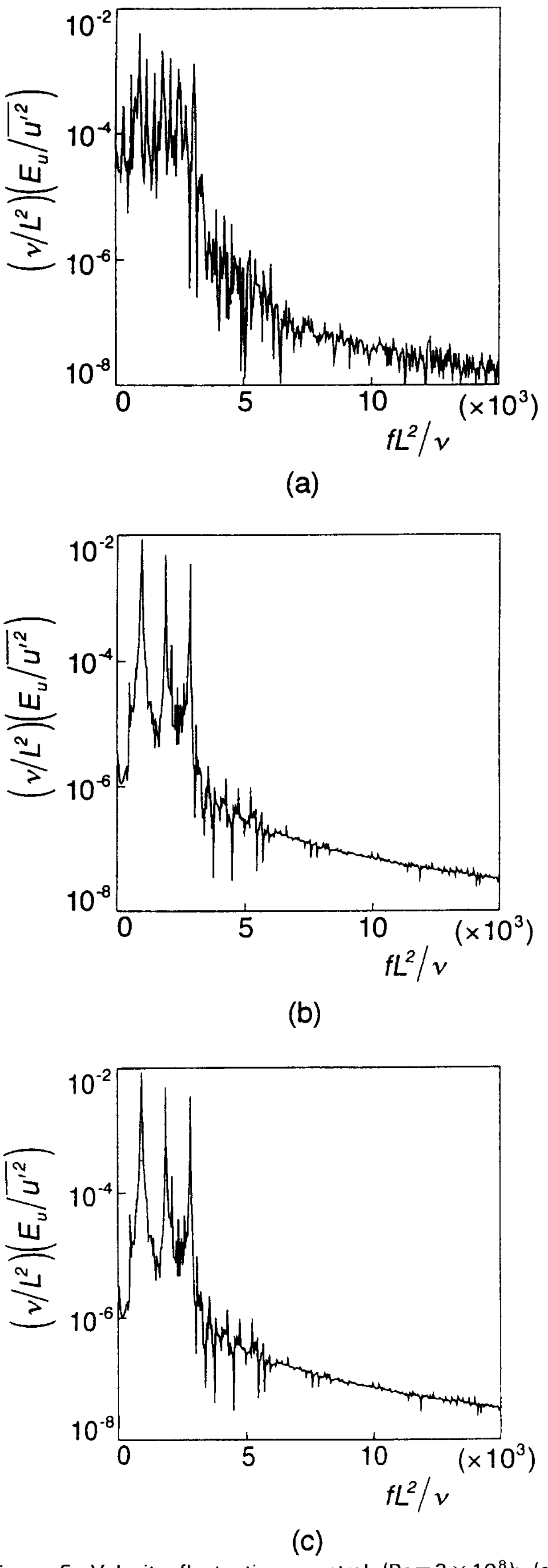

Figure 5 Velocity fluctuation spectral (Ra$=3\times10^8$); (a) Boussinesq; (b) LMN ($\beta\,\Delta T_w=0.095$); (c) exact ($\beta\,\Delta T_w=0.095$, $\gamma Ma^2=10^{-5}$)

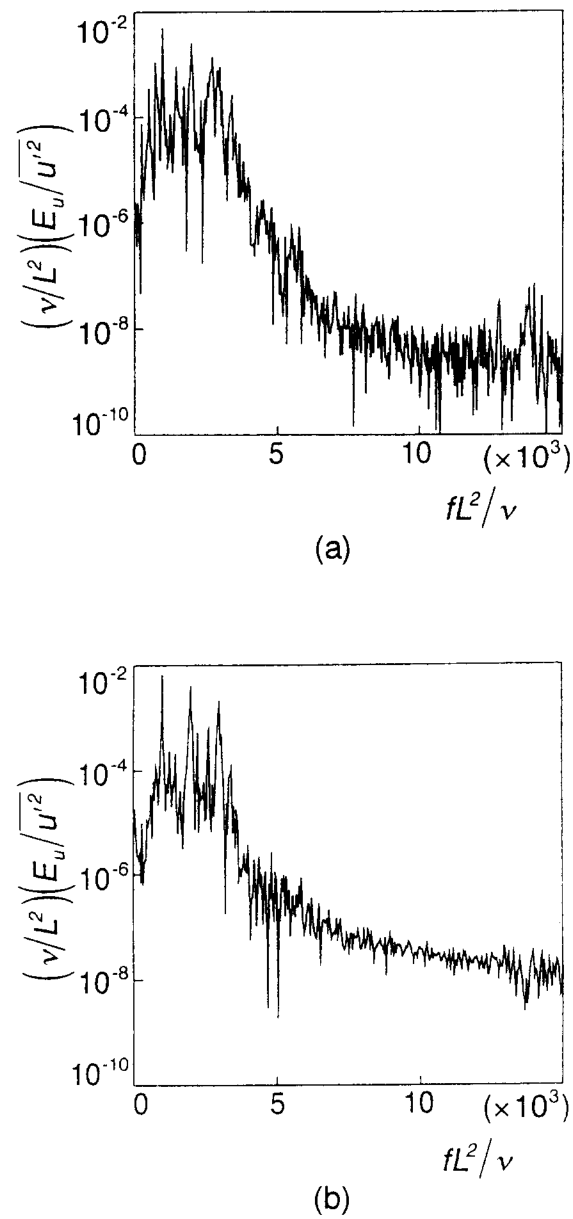

Figure 6 Velocity fluctuation spectra (Ra$=3.5\times10^8$, $\beta\,\Delta T_w=0.112$); (a) LMN; (b) exact ($\gamma Ma^2=10^{-5}$)

For Ra $= 3\times10^8$, the intensities of velocity and temperature fluctuations at half the cavity height are presented in Figure 8. A good agreement exists between the solutions of the low-Mach number and exact equations (Figures 8b and 8c). It is found that the fluctuations take their maximum intensity in the core region. On the contrary, the solution of the Boussinesq-approximated equations shows a different behavior (Figure 8a), where the intensity profiles of velocity and temperature fluctuations are symmetric and being flattened. Although these turbulent quantities obtained from the low-Mach number and exact equations coincide with each other up to Ra $= 3\times10^8$, some discrepancies appear at Ra $= 3.5\times10^8$, as seen in Figure 9. The intensities of velocity and temperature fluctuations become maximum near the boundary-layer edges. This indicates that the instability due to internal waves (Chenoweth and Paolucci 1986) or thermal effects (Ravi et al. 1994) occurs before the boundary layer becomes unstable. However, as expected from Figure 9, it is evident that

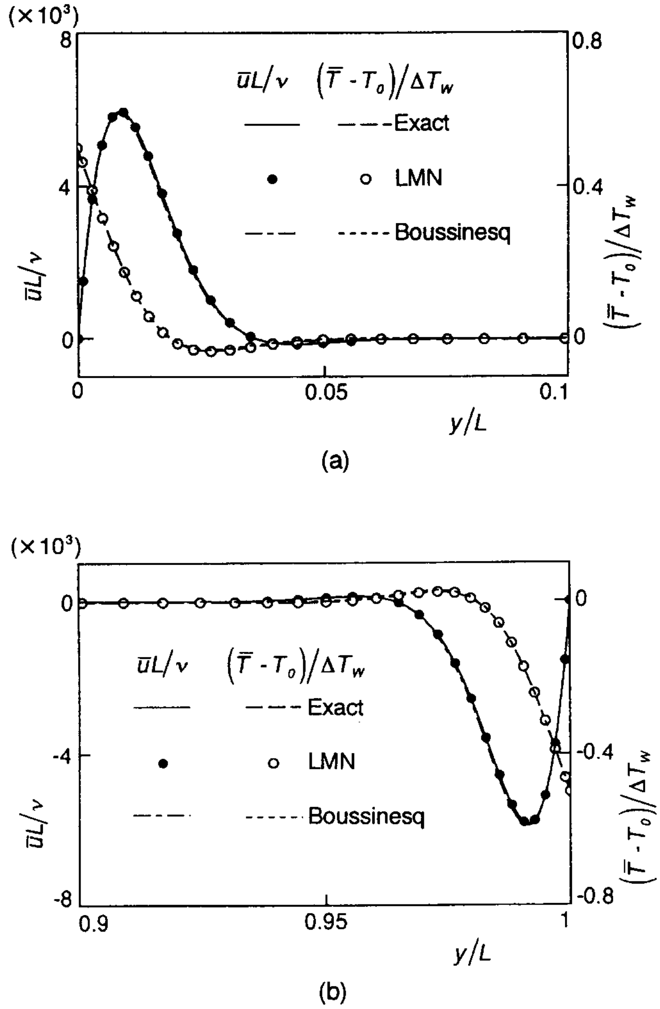

Figure 7 Profiles of mean vertical velocity and temperature at half the cavity height (Ra = 3.5 × 10^8); (a) hot vertical wall; (b) cold vertical wall

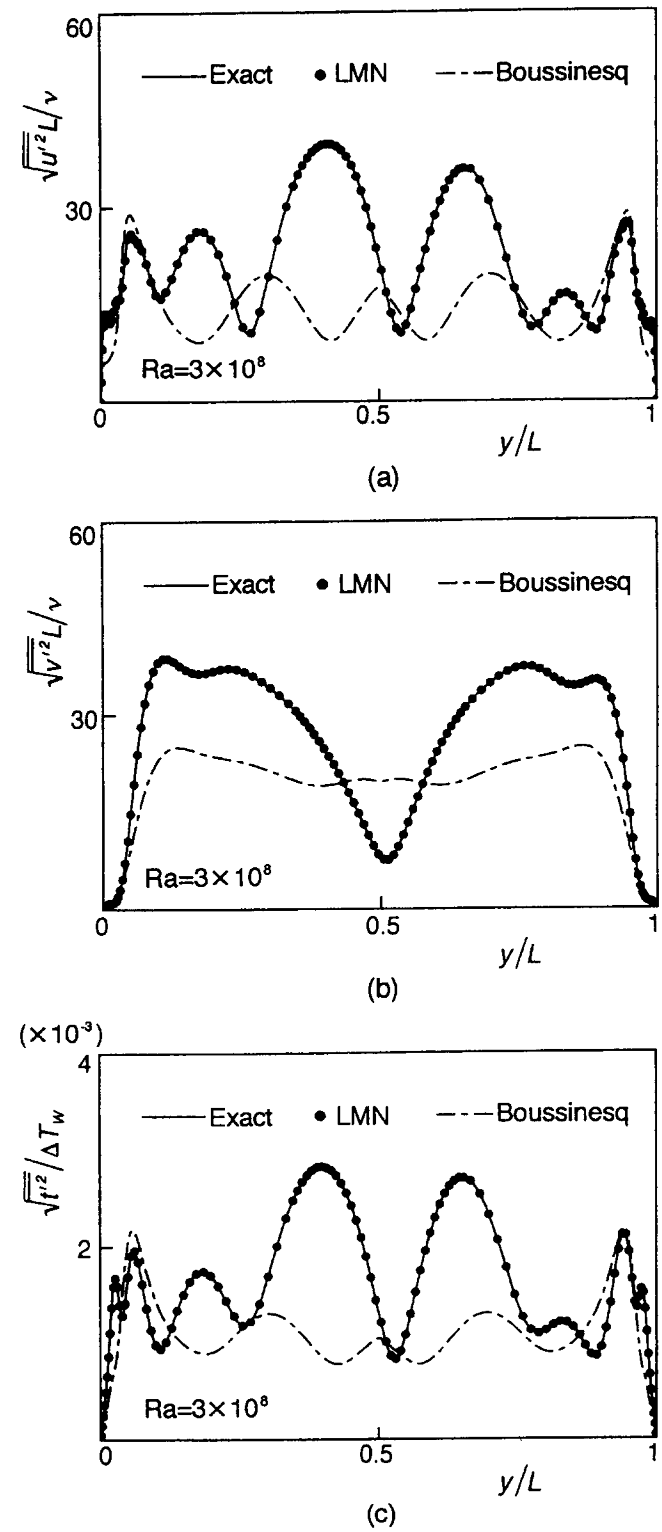

Figure 8 Intensities of velocity and temperature fluctuations at half the cavity height (Ra = 3 × 10^8); (a) vertical velocity; (b) horizontal velocity; (c) temperature

the transition to turbulence in thermal convection with increasing the Rayleigh number may not be correctly grasped, even with the low-Mach number approximation.

Concluding remarks

In general, laminar-turbulent transition of thermally driven flows in cavities has been discussed using approximated equations such as the Boussinesq-approximated and low-Mach number equations. By comparing the solutions obtained with the Boussinesq-approximated, low-Mach number, and exact equations (in a practical sense), it is found that the low-Mach number solutions certainly coincide with those of the exact equations up to a certain Rayleigh number. However, when the flow becomes weakly turbulent, the solutions given by the low-Mach number approximation deviate gradually from those of the exact equations. This means that even the low-Mach number equations, which are less restrictive than the Boussinesq-approximated equations, may not properly describe the transition behavior of the flow under realistic temperature conditions beyond a certain Rayleigh number. Therefore, the use of the exact governing equations may become necessary at higher-Rayleigh numbers. But these governing equations require large computational time

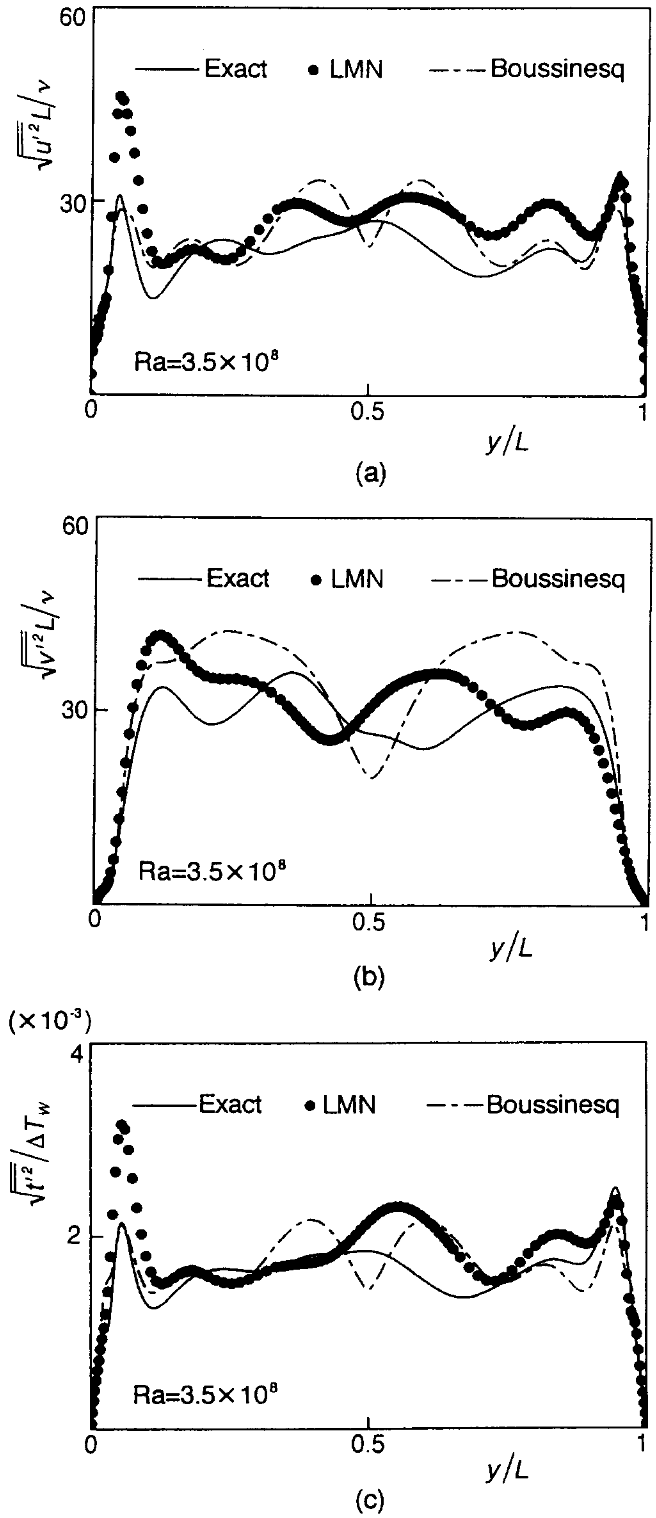

Figure 9 Intensities of velocity and temperature fluctuations at half the cavity height (Ra $= 3.5 \times 10^8$); (a) vertical velocity; (b) horizontal velocity; (c) temperature

for solving and thus are difficult to be developed into analysis of turbulence. A new approach would be needed for advanced studies of thermally driven flows.

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